Lecture 2 : Linear Algebra (1)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$$

$B_{n\times n}$ is a square matrix

$A_{m\times n} \pm B_{m\times n}$ ✓ Addition / Difference

their sizes are same

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3\times 4]$ × their sizes are not same!!

$A_{m\times n} \times B_{n\times k} = C_{m\times k}$

same

$A_{n\times n} \times B_{n\times n} = C_{n\times n}$ (square matrix)

ex $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}_{2\times 3}$

- $A_{2\times 2} \times B_{2\times 3}$ ✓
- $B_{2\times 3} \times A_{2\times 2}$ ×

$$AB = \begin{bmatrix} 2(4)+3(1) & 2(3)+3(-2) & 2(6)+3(3) \\ 1(4)+(-5)(1) & 1(3)+(-5)(-2) & 1(6)+(-5)(3) \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & -7 & -9 \end{bmatrix}$$

In general, $AB \neq BA$

Again, if A, B and AB are square matrix, $AB \neq BA$ in general.

ex $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 5(2)+(1)(4) & (5)(0)+(1)(3) \\ 3(2)+(-2)(4) & (3)(0)+(-2)(3) \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2(5)+(0)(3) & (2)(1)+(3)(-2) \\ 4(5)+(3)(3) & (4)(1)+(3)(-2) \end{bmatrix} = \begin{bmatrix} 10 & -4 \\ 29 & -2 \end{bmatrix} \neq AB$$

∴ $AB \neq BA$ is correct in general.

$A + B = B + A$

$A \times (B+C) = AB + AC$

$tA = At = \begin{bmatrix} ta_{11} & ta_{12} & \cdots \\ \vdots & & \\ ta_{m1} & ta_{m2} & \cdots \end{bmatrix}$, where t is a scalar

Note : $t + A$ is impossible because they are in different size.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$ Identity matrix

$A_{m\times n} I_{n\times n} = A_{m\times n} \Rightarrow AI = A$

$I_{m\times m} A_{m\times n} = A_{m\times n} \Rightarrow IA = A$

$\Big\}\ AI_1 = I_2 A = A$

$AI = IA \Rightarrow AD = DA$

$AA = A^2$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, it <u>does not</u> mean $A^2 = \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$. In fact, $A^2 \neq \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$.

$A^n = \overbrace{AA\cdots A}^{n}$

$A^0 = I$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}$ Transpose

$A_{1\times n}$ row vector

$A_{n\times 1}$ column vector

$[1\ \ 2] + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ We cannot deal with it !!

$[1\ \ 2] + [3\ \ 4]$ It can be calculated, Ans: $[4\ \ 6]$

$A_{1\times n} + B^T_{n\times 1}$

$(AB)^T = B^T A^T \Rightarrow A_{m\times n} B_{n\times k} = (B^T)_{k\times n} (A^T)_{n\times m}$

$k\times m \quad k\times m$

$\begin{cases} a_1 x + b_1 y = C_1 \\ a_2 x + b_2 y = C_2 \end{cases}$ x, y unknown

ex $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 6 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 6 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{①\times 4 + ③ \to ③} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 6 \\ 0 & 3 & 13 & -9 \end{array}\right]$

$\xrightarrow{-②/2 \to ②} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Then, we get $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 0 \\ x_3 = 0 \end{cases}$ $\xleftarrow{x_3 = 3}$

$x_1 = 6,\ x_2 = 3,\ x_3 = 0$

$\therefore \vec{x} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} = [6\ 3\ 0]^T$

ex $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

↳ zero row: the solution must not unique $\longrightarrow$ infinity numbers

$0 = 0$

$x_3 \in \mathbb{R}\ ;\ x_3 = \alpha \in \mathbb{R}$

ex $\left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\Rightarrow 0 \neq 1$ no solution

$A\vec{x} = \vec{b}$, $\vec{b} \neq 0$ (non-homogeneous)

$A\vec{x} = 0$ (homogeneous)

It can be divided as 3 cases

① 0 ② infinity solution ③ no solution

n unknowns, n equations

$A\vec{x} = 0$ only $\vec{x} = 0$

A is nonsingular

$A\vec{x} = b$ unique solution and $\vec{x}$ is something

$BA = I$, A is nonsingular

BA is I (known), $BA\vec{x} = B\vec{b}$

$I\vec{x} = B\vec{b}$

$\vec{x} = B\vec{b}$

$BA = I$, A is nonsingular $\Rightarrow AB = I$

A^{-1} : inverse of $A = B$

* $\boxed{AA^{-1} = A^{-1}A = I}$

A, B are non-singular (or invertible)

$\Rightarrow A^{-1}, B^{-1}$ exist.

$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

$(AB)(AB)^{-1} = I$

$ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$

$\boxed{(AB)^T = B^TA^T}$

$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_2A_1$, the order are also inversed

$\boxed{\begin{aligned}(A^{-1})^{-1} &= A \\ (tA)^{-1} &= \frac{1}{t}A^{-1}\end{aligned}}$

Prove: $tA() = I$

$At \times \frac{1}{t}A^{-1} = I$

$$\begin{bmatrix} 1 & & 1 \\ & 1 & \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & & 3 \\ & 2 & \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$E_5\cdots E_2E_1A\vec{x} = E_5\cdots E_2E_1\vec{b}$, where $E_5\cdots E_2E_1 = A^{-1}$

$E_5\cdots E_2E_1A\vec{x} \rightarrow I$, $E_5\cdots E_2E_1\vec{b} \rightarrow \vec{x}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}\right]$$

$[A \mid I]$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

$[I \mid A^{-1}]$

$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$ (use this technique to check your answers!)

Determinant

$\det(A) = |A|$ ← not absolute value

Cofactor $= (-1)^{i+j}M_{ij}$

$|A| = |A^T| = \sum_i^n a_{ik}C_{ik}$

C_{ik} ↓ cofactor

$$\begin{bmatrix} 1 & 6 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 6\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 5$$

$|(P)| = d_1d_2\cdots d_n$

(P) ↓ Diagonal

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & -2 \end{vmatrix}$$

- Interchanging any two columns (or any two rows) changes the sign of the determinant.
- The value of a determinant is not changed if we put its rows as columns, in the same consecutive order, i.e. $|A| = |A^T|$
- If a determinant has two rows (or two columns) identical, its value is zero
- A common factor of any row or column can be taken out to multiply the remaining determinant value
- If each element of a row (or a column) is the sum of two or more terms, the determinant can be expressed as the sum of two or more determinants

if $|A| \neq 0 \Leftrightarrow d_1 d_2 \cdots d_n \neq 0$

$A \to I$

* $|A| \neq 0 \Leftrightarrow$ A nonsingular

* $\Leftrightarrow A^{-1}A = I$
  $\Leftrightarrow A\vec{x} = 0$ only trivial solution $\vec{x} = 0$
  $\Leftrightarrow A\vec{x} = \vec{b}$ has unique solution